# Supercriticality for Annealed Approximations of Boolean Networks

Thomas Mountford [1]
Daniel Valesin [1]

June 7, 2010

**Abstract**

We consider a model proposed by Derrida and Pomeau (1986) and recently studied by Chatterjee and Durrett (2009); it is defined as an approximation to S. Kauffman's boolean networks (1969). The model starts with the choice of a random directed graph on $n$ vertices; each node has $r$ input nodes pointing at it. A discrete time threshold contact process is then considered on this graph: at each instant, each site has probability $q$ of choosing to receive input; if it does, and if at least one of its inputs were occupied by a 1 at the previous instant, then it is labeled with a 1; in all other cases, it is labeled with a 0. $r$ and $q$ are kept fixed and $n$ is taken to infinity. Improving a result of Chatterjee and Durrett, we show that if $qr > 1$, then the time of persistence of the dynamics is exponential in $n$.

*Keywords*: boolean networks, threshold contact process

[1]Institut de Mathématiques, Station 8, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland, daniel.valesin@epfl.ch

# 1 Introduction

In this paper we consider a threshold contact process on a random graph. Let $r \in \mathbb{N}$ and $q \in [0,1]$. For $n \in \mathbb{N}$, $n > r$, let $V_n = \{1, \dots, n\}$. For each $x \in V_n$, choose $r$ distinct points $y_1(x), \dots, y_r(x)$ in $V_n - \{x\}$; this choice is made uniformly among all $\frac{(n-1)!}{(n-r-1)!}$ possibilities and independently for each $x \in V_n$. Let $E_n = \{(y_i(x), x) : x \in V_n, 1 \leq i \leq r\}$ and call $G_n = (V_n, E_n)$ the graph thus obtained, a random directed graph on $n$ vertices and in-degree equal to $r$.

Once $G_n$ is chosen, it remains fixed and we consider a discrete time Markov chain with state space $\{0,1\}^{V_n}$ and initial configuration $\xi_0 \in \{0,1\}^{V_n}$, which will be deterministic for all our purposes. Let $\{B_t^x : x \in V_n, t \geq 1\}$ be a family of independent Bernoulli random variables with parameter $q$; given $\xi_t \in \{0,1\}^{V_n}$, we put

$$\xi_{t+1}(x) = \begin{cases} 1 & \text{if } B_{t+1}^x = 1 \text{ and } \sum_{i=1}^r \xi_t(y_i(x)) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

When $B_t^x = 1$, we say that $x$ *receives input* at time $t$; therefore, a vertex is set to 1 if and only if it receives input at that time and at least one of its input vertices $y_1(x), \dots, y_r(x)$ was set to 1 at the previous time. Given $A \subset V_n$, $\xi_t^A$ denotes the chain with initial configuration $\xi_0^A = \mathbb{1}_A$, the indicator function of $A$. We write $\xi_t^x$ instead of $\xi_t^{\{x\}}$. $\mathbb{P}_n$ will denote the probability measure both for the choice of $G_n$ and for the family $\{B_t^x\}$ (they are of course taken independently).

This setting was proposed by Derrida and Pomeau ([2]) as an "annealed approximation" to the less tractable S. Kauffman's boolean networks ([4]). These are cellular automata with rules of evolution randomly chosen, intended to model the interactions of genes in a cell. We refer the reader to [1] for the detailed scientific background, including the relationship between the threshold contact process defined above and Kauffman's original model.

For fixed $n$ and any initial configuration $\xi_0 \in \{0,1\}^{V_n}$, with probability one the threshold contact process eventually reaches the absorbing configuration in which all vertices are in state 0. The main object of investigation both in [2] and [1] is the distribution of this random time as a function of $n$, in particular as $n \to \infty$. Define $\rho = \rho(q, r)$ as the survival probability for a branching process in which individuals have probability $q$ of having $r$ children and $1 - q$ of having none. In [1] the following is proved.

**Theorem 1.1** *[Chatterjee and Durrett 2009]*
*(i.) If $qr > 1$, then for every $\eta > 0$ there exist $c > 0, b \in (0,1)$ such that, as $n \to \infty$,*

$$\inf_{0 \leq t \leq e^{cn^b}} \mathbb{P}_n \left( \frac{|\xi_t^{V_n}|}{n} \geq \rho - \eta \right) \to 1.$$

*(ii.) If $q(r-1) > 1$, then for every $\eta > 0$ there exists $c > 0$ such that, as $n \to \infty$,*

$$\inf_{0 \leq t \leq e^{cn}} \mathbb{P}_n \left( \frac{|\xi_t^{V_n}|}{n} \geq \rho - \eta \right) \to 1.$$

In this paper we improve this result.

**Theorem 1.2** *If $qr > 1$, then for every $\eta > 0$ there exists $c > 0$ such that, as $n \to \infty$,*

$$\inf_{0 \le t \le e^{cn}} \mathbb{P}_n \left( \frac{|\xi_t^{V_n}|}{n} \ge \rho - \eta \right) \to 1.$$

To explain why this result is to be expected and, in particular, the link with the mentioned branching process, we introduce the time dual of the model. Fix a realization of $G_n = (V_n, E_n)$ and $\{B_t^x : x \in V_n, t \ge 1\}$, define $\hat{E}_n$ as the set of directed edges obtained by inverting the edges of $E_n$ and $\hat{G}_n = (V_n, \hat{E}_n)$. Fix $T > 0$ and put $\hat{B}_t^{x,T} = B_{T-t}^x$ for $0 \le t < T$. Given $A \subset V_n$, define $\hat{\xi}_0^{A,T} = \mathbb{1}_A$ and, for $0 \le t < T$,

$$\hat{\xi}_{t+1}^{A,T}(x) = \begin{cases} 1 & \text{if for some } z, i, \text{ we have } y_i(z) = x,\ \hat{\xi}_t^{A,T}(z) = 1 \text{ and } \hat{B}_t^{z,T} = 1; \\ 0 & \text{otherwise.} \end{cases}$$

When $\hat{\xi}_t^T(z) = 1$ and $\hat{B}_t^{z,T} = 1$, we say that $z$ *gives birth* at time $t$, in which case $y_1(z), \dots, y_r(z)$ will all be in state 1 at $t+1$. We have the *duality equation*

$$\{\xi_T^A \cap B \ne \emptyset\} = \{\hat{\xi}_T^{B,T} \cap A \ne \emptyset\}$$

(we abuse notation associating $\xi \in \{0,1\}^{V_n}$ with $\{x \in V_n : \xi(x) = 1\}$). Since we will only work with the dual process, we will drop the superscript $T$ and assume that $\hat{\xi}_t^A$ is defined for all positive times with the evolution rule defined above.

Now, assume that $n$ is very large with respect to $r$. If $g$ is another integer that is much larger than $r$ and much smaller than $n$, then with high probability the set

$$\{z \in V_n : \text{ for some } k \le g \text{ and } z_1, \dots, z_k \in V_n, \text{ we have } x \to z_1 \to \dots \to z_k \to z \text{ in } \hat{G}_n\}$$

will simply be a directed tree of degree $r$ rooted in $x$. Conditioning on this event, the evolution of $|\hat{\xi}_t^x|$ up to time $g$ will be exactly that of the branching process mentioned before Theorem 1.1. In addition, it is not difficult to see that, without any conditioning, $|\hat{\xi}_t^x|$ is stochastically dominated by such a process. This remarks clarify why the model exhibits two phases in exact correspondence with the branching process. If the expected offspring size $qr \le 1$, then $\hat{\xi}_t^x$ dies out faster than the corresponding subcritical branching process, and the primal $\xi_t^{V_n}$ rapidly reaches the zero state. On the other hand, if $qr > 1$, the above theorem states that the system survives for a time that is exponentially large in $n$, characterizing the supercritical regime.

In the treatment of the dynamics, our proof is basically an exact repetition of that of [1]. What we do different is a more careful examination of the random graph. In order to argue that the confinement to a finite graph takes a long time to affect the dynamics, Chattterjee and Durrett prove an isoperimetric inequality that states that, if $m$ is small in relation to $n$, then with high probability there are no subsets $A \subset V_n$ of size $m$ such that the "influence set" $\{y_i(x) : 1 \le i \le r, x \in V_n\}$ has much less than $rm$ elements. We push this argument further and control the influence set along several generations rather than only the first one.

## 2 Proof of Theorem 1.2

In all the results and proofs in this section, we assume that $qr > 1$. Also, once and for all we fix $\tilde{q} < q$ such that $\tilde{q}r > 1$, $\delta < (\tilde{q}r - 1) \wedge 1$ and $g \in \mathbb{N}$ such that $(\tilde{q}r - 1 - \delta)(\tilde{q}r)^{g-1} > 1 + \delta$.

The following lemma is proved in [1]; see Lemma 2.2 and Equation (2.14) in that paper.

**Lemma 2.1** *For every $\eta > 0$ there exist $a > 0,\ b \in (0,1)$ such that, as $n \to \infty$,*

$$\mathbb{P}_n\left(\frac{|\{x : |\hat{\xi}^x_{\lceil a\log n\rceil}| > n^b\}|}{n} > \rho - \eta\right) \to 1.$$

We now introduce some definitions and notation. Given $m \in \mathbb{N}$, let

$$T^i_m = \{1,\dots,m\} \times \{1,\dots r\}^i, \quad 0 \le i \le g,$$

$$T_m = \cup_{i=0}^g T^i_m.$$

For $\sigma = (\sigma_0,\dots,\sigma_i), \sigma' = (\sigma'_0,\sigma'_1,\dots,\sigma'_j) \in T_m$, we say $\sigma \prec \sigma'$ either if $i < j$ or if $i = j$ and $\sigma$ is less than $\sigma'$ in lexicographic order. With this order, we can take an increasing enumeration $T_m = \{\sigma^1,\dots,\sigma^{(1+r+\dots+r^g)m}\}$. Then, $T^0_m = \{\sigma^1,\dots,\sigma^m\}$ and, for $i \ge 1$, $T^i_m = \{\sigma^{(1+r+\dots+r^{i-1})m+1},\dots,\sigma^{(1+r+\dots+r^i)m}\}$.

Next, we endow $T_m$ with directed edges by setting

$$\sigma \to \sigma' \text{ if and only if } \sigma = (\sigma_0,\dots,\sigma_i),\ \sigma' = (\sigma_0,\dots,\sigma_i,\sigma'_{i+1}) \text{ for some } i.$$

$T_m$ is thus the disjoint union of $m$ rooted, directed and $r$-regular trees, each with $g$ generations above the root. $\{0,1\}^{T_m}$ will be called the space of configurations. Given vertex $\sigma \in T_m$ and configuration $\psi \in \{0,1\}^{T_m}$, $\psi(\sigma) \in \{0,1\}$ will denote the value of $\psi$ at $\sigma$.

Let $A \subset V_n$ with $|A| = m$. We can enumerate $A = \{x_1,\dots x_m\}$ in the order of the indices of $V_n$. Given $\sigma = (\sigma_0,\dots,\sigma_i) \in T_m$ with $i > 0$, let $z^\sigma = y_{\sigma_i}(y_{\sigma_{i-1}}(\cdots(y_{\sigma_1}(x_{\sigma_0}))\cdots))$. Finally, define

$$\mathcal{A}^\sigma = \{z^{\sigma'} \in T_m : \sigma' \prec \sigma\}.$$

We now present an algorithm to construct a configuration $\psi = \psi(A) \in \{0,1\}^{T_m}$ from $A$.

**for** $j = 1$ **to** $m$ **set** $\psi(\sigma^j) = 0$;
**for** $j = m+1$ **to** $(1+r+\dots+r^g)m$
  **if** $\left[\psi((\sigma^j)_0,(\sigma^j)_1,\dots,(\sigma^j)_l) = 1 \text{ for some } l\right]$ **or** $\left[z^{\sigma^j} \notin \mathcal{A}^{\sigma^j}\right]$
    **then** set $\psi(\sigma^j) = 0$
  **else** set $\psi(\sigma^j) = 1$

In words, vertices are inspected in order; the roots are all set to 0 and the other vertices are set to 0 either if one of their ancestors has already been marked with a 1 or if their image under the map $\sigma \mapsto z^\sigma$ has never been seen before; otherwise they are set to 1.

**Lemma 2.2** *Given $A \subset V_n$ with $|A| = m$ and $\sigma^{i_1},\dots,\sigma^{i_k} \in T_m$,*

$$\mathbb{P}_n\big(\ [\psi(A)](\sigma^{i_1}) = \dots = [\psi(A)](\sigma^{i_k}) = 1\ \big) \le \left(\frac{m + rm + \dots + r^g m}{n-r}\right)^k.$$

*Proof.* There is no loss of generality in assuming that $\sigma^{i_a} \prec \sigma^{i_b}$ when $a < b$. We then have

$$\mathbb{P}_n\big(\ [\psi(A)](\sigma^{i_k}) = 1 \ \big|\ [\psi(A)](\sigma^{i_1}) = \dots = [\psi(A)](\sigma^{i_{k-1}}) = 1\ \big) \le \frac{m + rm + \dots + r^g m}{n-r}.$$

Indeed, let $\Theta^{i_k}$ denote the event that none of the ancestors of $\sigma^{i_k}$ in $T_m$ is marked with a 1 in $\psi(A)$. First note that $\{[\psi(A)](\sigma^{i_k}) = 1\} \subset \Theta^{i_k}$, because the algorithm fills all positions above a 1 with 0's. Next, fix $a_{m+1}, a_{m+2}, \ldots, a_{i_k-1} \in V_n$ such that

$$\{z^{\sigma^{m+1}} = a_{m+1}, \ldots, z^{\sigma^{i_k-1}} = a_{i_k-1}\} \subset \Theta^{i_k} \cap \{\ [\psi(A)](\sigma^{i_1}) = \ldots = [\psi(A)](\sigma^{i_k-1}) = 1\ \}$$

(we start at $m+1$ because $z^{\sigma^1}, \ldots, z^{\sigma^m}$ are deterministic, equal to the points of $A$). Then, conditioned on $\{z^{\sigma^{m+1}} = a_{m+1}, \ldots, z^{\sigma^{i_k-1}} = a_{i_k-1}\}$, there are at least $n-r$ possible positions for $z^{\sigma^{i_k}}$, and $[\psi(A)](z^{\sigma^{i_k}}) = 1$ precisely when $z^{\sigma^{i_k}} \in \mathcal{A}^{\sigma^{i_k}}$, a set of size less than $m + rm + \ldots + r^g m$. ∎

For $i < g$ and $B \subset T_m^i$, write $J(B) = \{\sigma' \in T_m : \sigma \to \sigma' \text{ for some } \sigma \in B\} \subset T_m^{i+1}$. Again for $i < g$, a set of sets $\mathcal{B} = (B_0, B_1, \ldots, B_i)$ with $B_0 \subset T_m^0$ and $B_{j+1} \subset J(B_j)\ \forall j \in \{1, \ldots, i-1\}$ will be called a *family* in $T_m$. Given $\psi \in \{0,1\}^{T_m}$, a family $\mathcal{B} = (B_0, \ldots, B_i)$ in $T_m$ is called *$\psi$-admissible* if

$(i.)$ $\cup_{j=0}^{i} B_j \subset \{\psi = 0\}$;

$(ii.)$ $|B_0| \geq \tilde{q}m$; $\forall j \in \{1, \ldots, i\}$, $|B_j| \geq \tilde{q} \cdot |J(B_{j-1}) \cap \{\psi = 0\}|$.

$\mathcal{B}$ will be called *$\psi$-good* if it is $\psi$-admissible and

$(iii.)$ $\forall j \in \{0, 1, \ldots, i\}$, $|J(B_j) \cap \{\psi = 0\}| \geq (\tilde{q}r - 1 - \delta)(\tilde{q}r)^j m$.

Finally, we say that $\psi$ is *robust* if all $\psi$-admissible families are $\psi$-good. The next lemma shows the motivation for these definitions.

**Lemma 2.3** *There exist $c, C > 0$ such that, for every $n$ and $A \subset V_n$,*

$$\mathbb{P}_n\left(|\hat{\xi}_g^A| < (1+\delta)|A| \;\middle|\; \psi(A) \text{ is robust}\right) \leq Ce^{-c|A|}.$$

*Proof.* We define inductively a family $\mathcal{B} = (B_0, \ldots, B_{g-1})$:

$$B_0 = \{\sigma \in T_{m,g}^0 : z^\sigma \text{ gives birth at time } 0\},$$

$$B_{j+1} = \{\sigma \in J(B_j) \cap \{\psi(A) = 0\} : z^\sigma \text{ gives birth at time } j+1\}.$$

The definition of $B_0$ implies that $\hat{\xi}_1^A \supset \{z^\sigma : \sigma \in J(B_0)\}$. From the construction of $\psi(A)$ we see that $\sigma \mapsto z^\sigma$ is injective on $J(B_0) \cap \{\psi(A) = 0\}$, so we have $|\hat{\xi}_1^A| \geq |J(B_0) \cap \{\psi(A) = 0\}|$. Iterating this argument we get

$$|\hat{\xi}_j^A| \geq |J(B_{j-1}) \cap \{\psi(A) = 0\}|, \quad 1 \leq j \leq g. \tag{2.1}$$

From this equation with $j = g$, property $(iii)$ in the definition of $\psi(A)$-good families and the choice of $g$ in the beginning of this section, we see that the result will follow from

$$\mathbb{P}_n\left(\mathcal{B} \text{ is not } \psi(A)\text{-admissible} \;\middle|\; \psi(A) \text{ is robust}\right) \leq Ce^{-c|A|} \qquad \forall n. \tag{2.2}$$

Define the events $G_0 = \{|B_0| < \tilde{q}m\}$, $G_j = \{|B_j| < \tilde{q} \cdot |J(B_{j-1}) \cap \{\psi(A) = 0\}|\}, 1 \leq j \leq g-1$. We now have $\{\psi(A) \text{ is robust}, \mathcal{B} \text{ is not } \psi(A)\text{-admissible}\} \subset \cup_{j=0}^{g-1}(\{\psi(A) \text{ is robust}\} \cap G_j)$, so

$$\mathbb{P}_n\left(\mathcal{B} \text{ is not } \psi(A)\text{-admissible} \;\middle|\; \psi(A) \text{ is robust}\right) \leq \mathbb{P}_n(G_0 \mid \psi(A) \text{ is robust}) + \sum_{j=1}^{g-1} \mathbb{P}_n\left(\, G_j \;\middle|\; \psi(A) \text{ is robust}, \cap_{l=0}^{j-1} G_l^c \,\right). \tag{2.3}$$

In order to bound the terms of the sum, we will need the estimate

$$x \in (0,1) \Longrightarrow \mathbb{P}(\ \mathrm{Bin}(k,p) < xkp\ ) \le \exp\{-\gamma(x)kp\}, \tag{2.4}$$

where $\gamma(x) = x\log x - x + 1$. This follows from Markov's inequality; see Lemma 2.3.3 in [3]. We start noting that

$$\mathbb{P}_n(G_0 \mid \psi(A) \text{ is robust}) = \mathbb{P}(\ \mathrm{Bin}(m,q) < \tilde{q}m\ ) \le \exp\{-\gamma(\tilde{q}/q)qm\}. \tag{2.5}$$

Next, on the event $\{\psi(A) \text{ is robust},\ \cap_{l=0}^{j-1} G_l^c\}$, the family $\mathcal{B}^j = (B_0, \ldots, B_{j-1})$ is $\psi(A)$-admissible, hence $\psi(A)$-good because $\psi(A)$ is robust, so $|J(B_{j-1}) \cap \{\psi(A) = 0\}| > (\tilde{q}r - 1 - \delta)(\tilde{q}r)^{j-1}m > (\tilde{q}r - 1 - \delta)m$. For $G_j$ to occur, less than $\tilde{q}|J(B_{j-1}) \cap \{\psi(A) = 0\}|$ can give birth at time $j$. Using (2.4), we get

$$\mathbb{P}_n\Big(\ G_j \mid \psi(A) \text{ is robust},\ \cap_{l=0}^{j-1} G_l^c\ \Big) \le \exp\{-\gamma(\tilde{q}/q)q(\tilde{q}r - 1 - \delta)m\}. \tag{2.6}$$

Putting (2.5) and (2.6) together back in (2.3), we get (2.2). ∎

**Lemma 2.4** *There exist $\epsilon, d, D > 0$ such that*

$$m \le \epsilon n \Rightarrow \mathbb{P}_n\left(\exists A \subset V_n : |A| = m,\ \psi(A) \textit{ is not robust}\right) \le De^{-dm} \qquad \forall n.$$

*Proof.* Fix $n \in \mathbb{N}$ and $A \subset V_n$ with $|A| = m$. Let $d_i = |\{\psi(A) = 1\} \cap T_m^i|$, $1 \le i \le g$ and $d = \sum d_i$. Let $i < g$ and $\mathcal{B} = (B_0, \ldots, B_i)$ be a $\psi(A)$-admissible family. We have

$$\begin{aligned}
&|B_0| \ge \tilde{q}m;\\
&|J(B_0) \cap \{\psi = 0\}| \ge \tilde{q}rm - d_1 \ge \tilde{q}rm - d;\\
&|B_1| \ge \tilde{q}^2rm - \tilde{q}d_1;\\
&|J(B_1) \cap \{\psi = 0\}| \ge (\tilde{q}r)^2m - \tilde{q}rd_1 - d_2 \ge (\tilde{q}r)^2m - \tilde{q}rd;\\
&\ldots\\
&|J(B_i) \cap \{\psi = 0\}| \ge (\tilde{q}r)^{i+1}m - (\tilde{q}r)^i d_1 - (\tilde{q}r)^{i-1}d_2 - \ldots - \tilde{q}rd_{i-1} - d_i\\
&\qquad\qquad\qquad\qquad \ge (\tilde{q}r)^{i+1}m - (\tilde{q}r)^i d.
\end{aligned}$$

Suppose that $d \le (1+\delta)m$. Then, for $j \in \{0, \ldots, i\}$ we have

$$|J(B_j) \cap \{\psi = 0\}| \ge (\tilde{q}r)^{j+1}m - (\tilde{q}r)^j d \ge (\tilde{q}r)^j(\tilde{q}r - 1 - \delta)m,$$

which means that $\mathcal{B}$ is $\psi(A)$-good. We have thus shown that $\{\psi(A) \text{ is not robust}\} \subset \{d > (1+\delta)m\}$. From this we get

$$\begin{aligned}
&\mathbb{P}_n\left(\exists A \subset V_n : |A| = m,\ \psi(A) \text{ is not robust}\right) \le \sum_{A:|A|=m} \mathbb{P}_n\left(\psi(A) \text{ is not robust}\right)\\
&\le \sum_{A:|A|=m}\ \sum_{d=\lceil(1+\delta)m\rceil}^{(1+r+\cdots+r^g)m}\ \sum_{D\subset T_m:|D|=d} \mathbb{P}_n\big([\psi(A)](\sigma) = 1\ \forall \sigma \in D\big).
\end{aligned}$$

We now bound $|\{D \subset T_m : |D| = d\}|$ by $2^{|T_m|}$ and use Lemma 2.2 to bound the probability; the above is then less than

$$\binom{n}{m} (1 + r + \cdots + r^g)\, m\, 2^{(1+r+\cdots+r^g)m} \left(\frac{(1+r+\cdots+r^g)m}{n-r}\right)^{(1+\delta)m}$$
$$\leq \left(\frac{ne}{m}\right)^m C^m \left(\frac{m}{n}\right)^{(1+\delta)m} \left(\frac{n}{n-r}\right)^{(1+\delta)m} \leq \left(C\left(\frac{m}{n}\right)^{\delta}\right)^m;$$

here $C$ is a constant that only depends on $r, g$ and $\delta$, and whose value has changed in the last inequality. Now it suffices to choose $\epsilon$ such that $C\epsilon^\delta < 1$. ∎

*Proof of Theorem 1.1.* From here on the proof continues very similarly to [1]; we present it for completeness. Fix $\eta > 0$,

- take $a, b$ corresponding to $\eta$ as in Lemma 2.1;
- take $c, C$ as in Lemma 2.3 and $\epsilon, d, D$ as in Lemma 2.4;
- define $F = C + D,\ f = c \wedge d,\ I = \lfloor e^{(f/2)n} \rfloor$;
- assume that $n$ is large enough so that $n^b < \epsilon n < I$ and $\delta n^b > 1$.

Define

$$s_i = \lceil a \log n \rceil + ig,$$
$$\alpha_i = (\lfloor n^b \rfloor + i) \wedge \lfloor \epsilon n \rfloor, \quad 0 \leq i \leq I.$$

Given $A \subset V_n$ and $\alpha \in \mathbb{N}$, let $\Pi_\alpha(A)$ denote the first $\alpha$ elements of $A$ (with respect to the order of $V_n$). For $x \in V_n$, define

$$\zeta_0^x = \Pi_{\alpha_0}\left(\hat{\xi}_{s_0}^x\right), \qquad \zeta_{i+1}^x = \Pi_{\alpha_{i+1}}\left(\hat{\xi}_{s_{i+1}}^{\zeta_i^x, s_i}\right),\ 0 \leq i \leq I,$$

where $\hat{\xi}_{s_{i+1}}^{\zeta_i^x, s_i}$ denotes the set of vertices that at time $s_{i+1}$ descend from $\zeta_i^x$ at time $s_i$. Finally, define the events

$$F_i^x = \{\psi(\zeta_i^x) \text{ is robust}\}, \quad G_i^x = \left\{\left|\hat{\xi}_{s_{i+1}}^{\zeta_i^x, s_i}\right| \geq (1+\delta)|\zeta_i^x|\right\}, \quad 0 \leq i \leq I;$$

$$H_{-1}^x = \{\hat{\xi}_{s_0}^x > n^b\}, \quad H_i^x = H_{i-1}^x \cap F_i^x \cap G_i^x, \quad 0 \leq i \leq I;$$
$$H = \cap_{x \in V_n}\left((H_{-1}^x)^c \cup H_I^x\right).$$

Fix $i \geq 0$ and assume that $H_i^x$ occurs. Then, $|\zeta_0^x| = \alpha_0$ and, by the definition of $G_0^x$, we have $|\hat{\xi}_{s_1}^{\zeta_0^x, s_0}| \geq (1+\delta)|\zeta_0^x| > |\zeta_0^x|$ because of the hypothesis $\delta n^b > 1$. So we have $|\zeta_1^x| = \alpha_1$, and arguing similarly we get $|\zeta_j^x| = \alpha_j$ for $1 \leq j \leq i+1$. Since $\zeta_j^x \subset \hat{\xi}_{s_j}^x$ for each $j$, we get $\left|\hat{\xi}_{\lceil a \log n \rceil + jg}^x\right| \geq |\zeta_j^x| = \alpha_j$ for each $j$. As a consequence,

$$H_I^x \subset \left\{\hat{\xi}_I^x \neq \emptyset\right\} \tag{2.7}$$

since $\lceil a \log n \rceil + (I+1)g \geq I$. We then have

$$\mathbb{P}_n\left(\left|\{x \in V_n : \hat{\xi}_I^x \neq \emptyset\}\right| > (\rho - \eta)n\right) \geq \mathbb{P}_n\left(H \cap \left\{\left|\{x \in V_n : |\hat{\xi}_{s_0}^x| > \alpha_0\}\right| > (\rho - \eta)n\right\}\right),$$

the reason being that, on the event in the second probability, at least $(\rho-\eta)n$ vertices reach time $s_0$ with more than $\alpha_0$ descendents, and all that do so continue having more than $\alpha_i$ descendents at times $s_i$, for all $i \le I$, and in particular are alive at time $I$, as seen in (2.7).

We know from Lemma 2.1 that, as $n \to \infty$,

$$\mathbb{P}_n\left(\left|\{x \in V_n : |\hat{\xi}^x_{s_0}| > \alpha_0\}\right| > (\rho-\eta)n\right) \to 1.$$

Using Lemmas 2.3 and 2.4, we also have

$$\begin{aligned}
\mathbb{P}_n(H^c) &\le \sum_{x\in V_n}\sum_{i=0}^{I}\left[\mathbb{P}_n(H^x_{i-1}\cap(F^x_i)^c)+\mathbb{P}_n(H^x_{i-1}\cap(G^x_i)^c)\right] \\
&\le \sum_{x\in V_n}\sum_{i=0}^{I}[\mathbb{P}_n(\exists A\subset V_n:\ |A|=\alpha_i,\ \psi(A) \text{ is not robust}) \\
&\qquad + \mathbb{P}_n\left(\left|\hat{\xi}^{\zeta^x_i,s_i}_{s_{i+1}}\right| < (1+\delta)|\zeta^x_i| \ \mid\ \psi(\zeta^x_i) \text{ is robust}\right)\Big] \\
&\le \sum_{x\in V_n}\sum_{i=0}^{I}[Ce^{-c\alpha_i}+De^{-d\alpha_i}] \le n\left((\epsilon n - n^b)Fe^{-f\lfloor n^b\rfloor} + (I-\epsilon n)Fe^{-f\lfloor \epsilon n\rfloor}\right)
\end{aligned}$$

which tends to 0 as $n \to \infty$. We thus have

$$\lim_{n\to\infty}\mathbb{P}_n\left(\left|\{x\in V_n : \hat{\xi}^x_I \neq \emptyset\}\right| > (\rho-\eta)n\right) = 1.$$

Using duality and atractiveness, we obtain

$$\inf_{0\le t\le I}\mathbb{P}_n\left(\frac{|\xi^{V_n}_t|}{n} \ge \rho-\eta\right) = \mathbb{P}_n\left(\frac{|\xi^{V_n}_I|}{n} \ge \rho-\eta\right) = \mathbb{P}_n\left(\frac{|\{x:\hat{\xi}^x_I\neq\emptyset\}|}{n} \ge \rho-\eta\right) \to 1,$$

completing the proof.

# References


[1] S. Chatterjee, R. Durrett, *Persistence of Activity in Threshold Contact Processes, an "Annealed Approximation" of Random Boolean Networks*, [arXiv:0911.5339v1].

[2] B. Derrida, Y. Pomeau, *Random networks of automata: a simple annealed approximation.*, Europhysics Letters 1 (1986), 45 - 49.

[3] R. Durrett, *Random Graph Dynamics*, Cambridge University Press.

[4] S. Kauffman, *Metabolic stability and epigenesis in randomly constructed genetic nets.* Journal of Theoretical Biology 22 (1969), 437 - 467.